Constructions and Rigid Transformations

In this unit, your student will be learning about constructing geometric figures. A *construction* in geometry class is similar to a construction site in the real world—students use a variety of materials to build something. At the beginning of the unit they only have two options: draw a line or draw a circle. It seems like that's not enough to make much, but this image is made entirely of circles:



Can you see how to add lines to make a triangle, rectangle, or hexagon?

In this unit, students also revisit some ideas first encountered in previous grades: *rotation*, *reflection*, and *translation*, which are the three *rigid transformations*. You might invite your student to look for transformations and *symmetry* in their everyday life.

What do you see in these two fences?



Each fence has a vertical line of reflection, because if you folded it in half, the left and right halves would match up. The chain-link fence also has a horizontal line of reflection, because if you folded it in half the other way, the top and bottom halves would match up. The picket fence doesn't have any rotational symmetry, but you could rotate the whole image of the chain-link fence 180 degrees and it would look the same.



Students are developing skills in proving their claims during this unit. So instead of saying "the fence looks symmetric," students would use the definition of reflection to show that every part of the left half lines up exactly with every part of the right half.

Here is a task to try with your student:

Line *AD* intersects line *EC* at point *B*, and *B* is the center of the circle. It may be helpful to draw on a piece of wax paper to see these moves.

Determine whether each statement is true or false. Explain how you know.

- 1. Reflect point E over the line AD. The image is point C.
- 2. Rotate point *C* 180 degrees clockwise using center *B*. The image is point *E*.
- 3. Rotate point D counterclockwise using center B and angle DBC. The image is point C.
- 4. Translate point *A* by the directed line segment *BD*. The image is point *B*.
- 5. Angle *ABE* is congruent to angle *DBC*.





- 1. False. The line connecting a point to its image has to be perpendicular to the line of reflection.
- 2. True. A 180-degree rotation takes C to a point on the other side of line BC, which is the same distance away from the center.
- 3. True. The path of the rotation will follow the edge of the circle.
- 4. False. The distance from *A* to *B* is not the same as the distance from *B* to *D*.
- 5. True. Rotating angle ABE 180 degrees using center B would take it to angle DBC, because when you rotate a line 180 degrees, it lands on itself. Rotation does not change the size of an angle.

Congruence

In this unit, your student will be learning about triangles and proof. Triangles are the building blocks of geometric figures. Once students understand triangles, they can apply their understanding to quadrilaterals and other shapes.

Students start out with some experiments. You can recreate these experiments at home with different-sized pieces of linguine.

- If I know 2 side lengths, is that enough to describe a unique triangle?
- How about 3 side lengths?
- If I know 2 side lengths, does that describe a unique quadrilateral?
- How about a unique rectangle?

If a set of information seems to work, make a *conjecture*. One conjecture is: 3 side lengths describes a unique triangle. In other words, if 2 triangles have all 3 sides of the same length, then one triangle fits exactly on top of the other. Any pair of figures (such as segments or triangles) in which we can find transformations that take one figure exactly onto the other figure so every part lines up is called *congruent*. So it seems that one way to create 2 triangles that are congruent is to have all 3 pairs of sides congruent. We can try dozen of triangles, and the triangles always seem to fit on top of each other exactly (even the angles!), but how can we be certain that it will work for every possible triangle anyone could ever make? For that, we need a proof that relies on precise definitions.

Proof is how mathematicians take a conjecture, a claim that seems to be true, and turn it into a theorem, a claim we are certain is true. To prove that something is true, every statement must be backed up with a reason. Students are building a list of reasons they can use for proofs in a reference chart. This list includes definitions, assumptions, and theorems they have already proven. Proofs in geometry work like court cases in which lawyers use evidence and case law to make an argument. They also work like arguments at home. Next time your student says you need to buy them something, ask them to prove it. They could use the definition of need and provide convincing evidence of that need, or they might have to adjust their conjecture and provide convincing evidence they deserve something they want instead.





Here is a task to try with your student:

- 1. Write a triangle congruence statement based on the diagram.
- 2. What information do you know that could help you write a proof?
- 3. Prove the triangles are congruent.
- 4. What type of quadrilateral does *ABDC* have to be?
- 5. What type of quadrilateral could *ABDC* possibly be?

- 1. Triangle *ABC* is congruent to triangle *DBC*. (Other orders such as $\triangle BAC \cong \triangle BDC$ are okay, but the corresponding letters have to match, so $\triangle ABC \cong \triangle BDC$ is not okay.)
- 2. $\overline{AC} \cong \overline{DC}$, because they're marked on the diagram. $\overline{AB} \cong \overline{DB}$, because they're both radii of the same circle.
- 3. It is given that sides *AC* and *DC* are congruent. Sides *AB* and *DB* are congruent because they're both radii of the same circle. Side *BC* is congruent to side *BC*, because they are the same segment. All 3 pairs of corresponding sides are congruent in triangles *ABC* and *DBC*, so the triangles are congruent by the SSS Triangle Congruence Theorem.
- 4. *ABDC* has to be a kite since it has 2 pairs of congruent sides and the congruent sides are next to each other.
- 5. *ABDC* could be a rhombus if *AC* and *DC* are the same length as the radii of the circle.

Similarity

In this unit, your student will be learning about similarity. They study a variety of similar figures and continue to write proofs about triangles. Then they use the statements they've proven to solve new problems.

Students start out with some comparisons. They look at different images to decide what stays the same and what changes with a scaled image. Imagine that you want to make a poster of a picture of a robot.

- Which image is a scaled copy of Image A?
- What happens to the shapes in the scaled copy?
- What happens to the angles in the scaled copy?
- What happens to the segments in the scaled copy?



It looks like some parts of the shape stay the same no matter what. The rectangles stay rectangles in all 3 images. But in Image B, the sides of the rectangle for the head look almost the same. It might even be a square. That isn't a scaled copy of the original Image A. The triangles for the legs in the original are twice as tall as they are wide. This same ratio holds for Image C. The proportionality of corresponding sides is one of the characteristics of a scaled copy. Another characteristic of a scaled copy is that the corresponding angles stay the same.



Recall that figures are called congruent if we can find rigid transformations (translation, rotation, reflection) that take one figure exactly onto the other figure so every part lines up. Two figures are called similar if we can find any transformations (translation, rotation, reflection, dilation) that take one figure exactly onto the other figure so every part lines up. The new transformation, dilation, makes scaled copies of figures.

For the robots, Image C is a translation and dilation of Image A. To dilate an image we need to choose a scale factor. The scale factor to go from the original to the larger size is 2. Every segment will be twice as long after the dilation. The scale factor to go from a standard photo to a wallet size photo would be something less than 1, such as $\frac{1}{2}$. The new image would be smaller, but all the angle measures stay the same and the ratios of the side lengths do too, so the image is not distorted.

Here is a task to try with your student:



Triangles *XYZ* and *LMN* are similar triangles.

- 1. Redraw the triangles so the corresponding sides are easier to see. Name the corresponding sides and angles.
- 2. Angle X is 45 degrees and angle N is 101 degrees. What are the measures of the other angles?
- 3. Side XY is 5 units long and side LM is 3 units long.
 - a. What is the scale factor of the dilation that takes triangle XYZ to triangle LMN?
 - b. What is the scale factor of the dilation that takes triangle LMN to triangle XYZ?





- Angle X is corresponding to angle L.
 Angle Y is corresponding to angle M.
 Angle Z is corresponding to angle N.
 Side XY is corresponding to side LM.
 Side YZ is corresponding to side MN.
 Side ZX is corresponding to side NL.
- 2. Angle $L = 45^{\circ}$. Angle $Z = 101^{\circ}$. Angle $M = Y = 34^{\circ}$.

a.
$$\frac{3}{5} = 0.6$$

b. $\frac{5}{3}$

Right Triangle Trigonometry

In this unit, your student will be learning about right triangle trigonometry. Trigonometry is the study of triangle measure. In a previous unit students studied similar triangles, now they can apply what they learned about similar triangles to right triangles in this unit. Right triangles turn out to be useful enough that there is a whole unit of study on them.



What do you notice about these triangles? What do you wonder about them?

You may notice that the hypotenuse (the longest side) is always twice as long as the shortest side. This ratio of 1 : 2 for short : hypotenuse applies to any triangle with angles measuring 30°, 60°, and 90°. That's because all of these triangles are similar triangles, and corresponding sides are proportional in similar triangles. The shortest side is opposite the 30 degree angle, so we call this ratio $sin(30) = \frac{1}{2}$. We say the sine of a 30 degree angle is equal to $\frac{1}{2}$. The definition of sine is the ratio of the opposite side to the hypotenuse in a right triangle.

С

Mathematicians recorded the ratios for right triangles with a variety of acute angles in tables. Then as calculators became more powerful the information in the table was programmed into scientific calculators. So instead of having to draw and measure the sides of a triangle, we can look up the ratio for any right triangle. This allows us to make calculations about triangle measures without making precise diagrams.

In this unit students learn the names of 3 trigonometric ratios. θ is a Greek letter used to represent an angle measure, such as 30 degrees in the previous example.



Here is a task to try with your student:

angle	adjacent leg ÷ hypotenuse	opposite leg ÷ hypotenuse	opposite leg ÷ adjacent leg
30°	0.866	0.500	0.577
40°	0.766	0.643	0.839
50°	0.643	0.766	1.192
60°	0.500	0.866	1.732

- 1. How long is side *AB*? Show or explain your reasoning.
- 2. How long is side *AC*? Show or explain your reasoning.
- 3. How long is side *DE*? Show or explain your reasoning.
- 4. How long is side *FD*? Show or explain your reasoning.





- 1. AB = 5 inches. It's half of 10 inches. $sin(30) = \frac{AB}{10}$ so $0.5 = \frac{AB}{10}$
- 2. $AC = \sqrt{75}$ or about 8.66 inches. $5^2 + (AC)^2 = 10^2$ so $AC = \sqrt{75}$ $\cos(30) = \frac{AC}{10}$ so $0.866 = \frac{AC}{10}$
- 3. DE = 64.3 feet. $sin(40) = \frac{DE}{100}$ so $0.643 = \frac{DE}{100}$

4.
$$FD = 76.6$$
 feet.
 $6.43^2 + (FD)^2 = 100^2$
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 $\cos(40) = \frac{FD}{100}$ so $0.766 = \frac{FD}{100}$

Solid Geometry

In this unit, your student will analyze properties of geometric solids. Since we live in three-dimensional space, people often need to solve problems about such solids. For example, a designer might need to create packaging for a candy bar in the shape of a triangular prism. An engineer might need to design a controller for a water tank in the shape of a cylinder. Or a lighting director for a theater might model the light from a spotlight using the shape of a cone.

When working with solids, we often need to visualize cross sections, or intersections between the solid and a plane. Here are all the kinds of cross sections we can find in a cylinder.



To find the volume of any prism or cylinder, no matter the shape of the base or if the figure is upright or oblique (slanted sideways), multiply the area of the base by the solid's height. This idea is captured in the formula V = Bh, where V is the volume, B is the area of the base, and h is the solid's height. For example, to find the volume of this cylinder, first calculate the area of the circular base using the expression πr^2 where r is the length of the base's radius. The base has area 16π square feet because $\pi(4)^2 = 16\pi$. Now we can conclude that the volume of the cylinder is 80π cubic feet because $16\pi \cdot 5 = 80\pi$.



The process to find the volume of a pyramid or cone is the same as for prisms and cylinders except that the result must be multiplied by $\frac{1}{3}$. That is, for pyramids and cones, $V = \frac{1}{3}Bh$.



For example, to find the volume of this rectangular pyramid, start by calculating the area of the base, which is 45 square centimeters because $5 \cdot 9 = 45$. Now substitute 45 and 12 into the volume formula to find that the volume of the pyramid is 180 cubic centimeters:

$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3} \cdot 45 \cdot 1$$
$$V = 180$$

2



Here is a task to try with your student:

Here is a cone.



- 1. One measurement that you need to calculate the volume is missing. Find the value of this measurement.
- 2. Calculate the volume of the solid.

- 1. The length of the radius is missing. Because this is a right triangle, the Pythagorean Theorem applies. One of the triangle's legs measures 24 inches and the hypotenuse measures 30 inches, so $24^2 + r^2 = 30^2$. Squaring the 24 and the 30, we get $576 + r^2 = 900$. Subtract 576 from both sides to get $r^2 = 324$. Now r is the positive number that squares to get 324, so the radius measures 18 inches because $\sqrt{324} = 18$.
- 2. The formula for the volume of a cone is $V = \frac{1}{3}Bh$. The cone's base is a circle with radius 18 inches. The area of the base is 324π square inches because $\pi(18)^2 = 324\pi$. Substitute this area and the cone's height of 24 inches into the volume formula to find that the volume of the cone is $2,592\pi$ cubic inches: $V = \frac{1}{3}Bh$

$$V = \frac{1}{3} \cdot 324\pi \cdot 24$$

$$V = 2,592\pi$$

Coordinate Geometry

In this unit, your student will make connections between geometry and algebra by working in the coordinate plane with geometric concepts from prior units. The coordinate grid imposes a structure that can provide new insights into ideas students have previously explored.

Your student has already worked with transformations. Here, they'll think about transformations as functions that take points in the plane as inputs and give other points as outputs. For example, the notation $(x, y) \rightarrow (x + 4, y - 2)$ means that to find the image for each point in a figure, we add 4 units to the *x*-coordinate and subtract 2 units from the *y*-coordinate. Let's apply this transformation to triangle *ABC*.

(<i>x</i> , <i>y</i>)	(x+4, y-2)
A : (-4, 1)	A':(0,-1)
<i>B</i> : (0, 2)	B':(4,0)
C: (-3, 3)	C':(1,1)



This transformation was a translation by the directed line segment from (-4, 1) to (0, -1), or informally, a translation 4 units right and 2 units down.

Transformations can also be used to analyze slopes of parallel and perpendicular lines. Suppose we draw a line passing through the point P = (-3, 2) and the point (0, 0), then apply the transformation $(x, y) \rightarrow (y, -x)$ to the line.





This rule rotates the line 90 degrees clockwise using the point (0, 0) as a center. The center of rotation doesn't move, so (0, 0) maps to itself. The image of point P is P' = (2, 3). The slope of the original line is $-\frac{2}{3}$, and the slope of the image is $\frac{3}{2}$. The slopes are *opposite reciprocals* of one another. Your student will use this to prove that *any* two perpendicular lines that aren't horizontal and vertical have slopes that are opposite reciprocals.

The Pythagorean Theorem proves useful in the coordinate plane as well. Consider the circle with center (2, 5) and radius 8 units. The point (6, 12) appears to be on the circle. We can test if it really is on the circle by calculating the distance between this point and the center. Start by drawing a right triangle whose hypotenuse is the distance between the 2 points.



The lengths of the triangle's legs can be calculated by subtracting the coordinates of the points: The vertical leg is 7 units long, and the horizontal leg is 4 units long. Substitute these into the Pythagorean Theorem.

$$a2 + b2 = c2$$
$$42 + 72 = c2$$

 $65 = c^2$

The distance between the points is the positive number that squares to make 65, or about 8.1 units. So, because it's not exactly 8 units away from the circle's center, the point (6, 12) isn't actually on the circle.



Here is a task to try with your student:

The image shows triangle J.



Apply each transformation rule to triangle *DEF*. Then, describe the transformation, and decide whether it produced a congruent image, a similar image, or neither.

- 1. Label the result of this transformation $M: (x, y) \rightarrow (-x, y + 5)$
- 2. Label the result of this transformation $N: (x, y) \rightarrow (3x, y)$



- 1. This transformation was a reflection across the *y*-axis, then a translation by the directed line segment from (-1, 0) to (-1, 5). All 3 corresponding pairs of sides of the original and image triangles are congruent, so the 2 triangles are congruent (and therefore also similar) by the Side-Side-Side Triangle Congruence Theorem. This makes sense because reflections and translations are rigid motions.
- 2. This transformation was a horizontal stretch away from the *y*-axis by a factor of 3. The corresponding vertical sides of triangle J and triangle N are congruent, but the horizontal side of triangle N is 3 times as long as the corresponding side in triangle J. Since pairs of corresponding sides are neither all congruent nor all proportional, the 2 triangles are neither congruent nor similar.

Circles

In this unit, your student will study properties of circles. Students start by exploring new vocabulary. In prior units, students worked with radii and diameters of circles. Here, several new concepts are defined: Chords are segments whose endpoints are on a circle. A line tangent to a circle intersects the circle in exactly one point. An arc is a portion of a circle's circumference between 2 endpoints.

arc



There are also some special angles defined in circles: A central angle is formed by 2 radii, and an inscribed angle is formed by 2 chords that share an endpoint. Your student will identify relationships between chords, tangent lines, arcs, central angles, and inscribed angles. For example, if an inscribed angle and a central angle define the same arc, then the measure of the inscribed angle is half that of the central angle. In the image, angle DCB is an inscribed angle, and its measure is half the measure of the corresponding central angle DAB.



Next, students examine inscribed and circumscribed circles. A circle is said to be circumscribed about a polygon if it passes through each of the polygon's vertices, and it's referred to as an inscribed circle if it is tangent to all sides of the polygon.

All triangles have both circumscribed and inscribed circles. To draw a circumscribed circle for a triangle, construct the perpendicular bisectors of the triangle's sides. These 3 lines meet at a point called the triangle's circumcenter. A circle centered at this point, with radius set to the distance between the circumcenter and a vertex of the triangle, will pass through all the triangle's vertices. To draw a triangle's inscribed circle, construct the triangle's angle bisectors, which all meet at a point called the incenter. The inscribed circle is centered at the incenter, and its radius is the distance from the incenter to any of the triangle's sides.





Your student will also study portions of circles. A sector is the region of a circle enclosed between two radii. To find the area of the sector in the image, first calculate the area of the full circle. This area is 900π square centimeters because $\pi(30)^2 = 900\pi$. The sector makes up $\frac{1}{6}$ of the circle because $\frac{60}{360} = \frac{1}{6}$. Multiply this fraction by the total area to find that the area of the sector is 150π square centimeters.



Finally, students have previously measured angles using degrees, but here they learn a new way to measure angles. The radian measure of an angle whose vertex is at the center of a circle is the ratio of the length of the arc defined by the angle to the radius of the circle. That is, $\theta = \frac{\text{arc length}}{\text{radius}}$. Radian measure will be useful when students study trigonometry in future courses.

Here is a task to try with your student:

A farmer has a circular field, created by a watering system that rotates around a center pivot point. The field's radius measures 400 meters. As shown in the image, part of the field is planted with oats and part is planted with wheat.



- 1. Find the area of the field that is planted with oats.
- 2. A road runs around the circumference of the circle. Find the length of the arc of the road defined by the wheat part of the field.

- 1. The total area of the field is $160,000\pi$ square meters because $\pi (400)^2 = 160,000\pi$. The 135 degree sector represents $\frac{3}{8}$ of the field because $\frac{135}{360} = \frac{3}{8}$. Multiply $160,000\pi$ by $\frac{3}{8}$ to find an area of $60,000\pi$ square meters of oats.
- 2. The total circumference of the field is 800π meters because $2 \cdot \pi \cdot 400 = 800\pi$. The wheat sector takes up $\frac{5}{8}$ of the field because $1 \frac{3}{8} = \frac{5}{8}$. Multiply 800π by $\frac{5}{8}$ to find that this portion of the road is 500π or about 1,571 meters long.

Conditional Probability

In this unit, your student will build on their understanding of probability, including conditional probability. The probability of an event is a number that measures how likely the event is to happen. It can be 0, 1, or any number in between. It is 0 if the event will never happen and 1 if the event must happen. If an event occurs half of the time in the long run, then its probability is 0.5. Conditional probability is the probability that one event occurs under the condition that another event occurs.

Here is an example. The table summarizes the type (medium, large, or extra-large) and condition (no cracked eggs, or one or more cracked eggs) of 50 cartons of eggs at a grocery store.

	medium	large	extra-large	total
one or more eggs cracked	1	3	1	10
no cracked eggs	4	22	19	40
total	5	25	20	50

One carton is selected at random.

What is the probability that the carton has no cracked eggs? This probability is 0.8. This is because 40 cartons have no cracked eggs out of a total of 50 cartons and $\frac{40}{50} = 0.8$. Students also see this type of of question written as P(no cracked eggs) which means "the probability that a randomly selected carton has no cracked eggs." In this case, P(no cracked eggs) = 0.8.

What is the probability that the carton has no cracked eggs under the condition that it is a carton of extra-large eggs? This conditional probability is 0.95. This is because 19 cartons of extra-large eggs had no cracked eggs out of a total of 20 cartons of extra-large eggs and $\frac{19}{20} = 0.95$. This type of probability is called conditional probability because it is a probability based on the condition of selecting a carton of extra-large eggs. Students see this type of question written as P(no cracked eggs) which means that the "probability that a randomly selected carton has no cracked eggs under the condition that it is a carton of extra-large eggs." In this case P(no cracked eggs | extra-large) = 0.95.

Here is a task to try with your student:

The table summarizes the position of loaves of bread at the grocery store (bread in the front row or bread not in the front row) and the sell-by date (within five days or more than 5 days away).

A loaf of bread is selected at random.

	sell-by date within 5 days	sell-by date more than 5 days away
bread in the front row	36	14
bread not in the front row	24	76

1. What is the probability that the bread has a sell-by date within 5 days?

- 2. What is the probability that bread has a sell-by date within 5 days under the condition that the loaf of bread is in the front row?
- 3. What is *P*(sell-by date more than 5 days away | bread not in the front row)?
- 4. You are in a rush and want to grab a loaf of bread at this store without looking at the sell-by date. Does grabbing the loaf of bread from the front row give you the best chance of getting a loaf of bread with a sell-by date more than 5 days away? Use probability to explain your reasoning.

- 1. 0.4 or $\frac{60}{150}$
- 2. 0.72 or $\frac{36}{50}$
- 3. 0.76 or $\frac{76}{100}$
- 4. No it does not give you the best chance of getting a loaf of bread with a sell-by date more than 5 days away. The probability of getting a loaf or bread with a sell-by date more than 5 days away under the condition that is in the front row is 0.28 compared to a probability of 0.72 for a loaf of bread that is not in the front row.